Universality for the breakup of invariant tori in Hamiltonian flows

[1]C. Chandre, [1]M. Govin, [1]H. R. Jauslin, and [2]H. Koch
[1]*Laboratoire de Physique, CNRS, Université de Bourgogne, BP 400, F-21011 Dijon, France*
[2]*Department of Mathematics, The University of Texas at Austin, Austin, TX 78712*

In this article, we describe a new renormalization-group scheme for analyzing the breakup of invariant tori for Hamiltonian systems with two degrees of freedom. The transformation, which acts on Hamiltonians that are quadratic in the action variables, combines a rescaling of phase space and a partial elimination of irrelevant (non-resonant) frequencies. It is implemented numerically for the case applying to golden invariant tori. We find a nontrivial fixed point and compute the corresponding scaling and critical indices. If one compares flows to maps in the canonical way, our results are consistent with existing data on the breakup of golden invariant circles for area-preserving maps.

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I. INTRODUCTION

In Hamiltonian systems with two (or more) degrees of freedom, smooth invariant tori typically persist under small perturbations [1–4]. The most stable tori appear to be the ones for which the frequency ratio is a quadratic irrational, such as the golden mean. As the strength of the perturbation passes some critical value, these tori are observed to exhibit self-similar scaling behavior, and then they break up [5,6]. To study such critical behavior, several renormalization schemes have been proposed [7–13]. The idea is to use a transformation $\mathcal{R}$ that maps a Hamiltonian H into a rescaled Hamiltonian $\mathcal{R}(H)$, in such a way that irrelevant degrees of freedom are contracted. The transformation $\mathcal{R}$ should have roughly the following properties: $\mathcal{R}$ has an attractive integrable fixed point H_0 that has a smooth invariant torus of a given frequency. Every Hamiltonian in its domain of attraction $\mathcal{D}$ has a smooth invariant torus of the desired frequency. Another non-integrable fixed point H_* lies on the boundary of $\mathcal{D}$, also called critical surface, and this nontrivial fixed point attracts the Hamiltonians on the critical surface. Figure 1 shows schematically the expected nature of the renormalization flow in the space of Hamiltonians. The existence of a nontrivial fixed point has strong implications concerning universal properties associated with the breakup of invariant tori. The analysis of the renormalization for area-preserving maps [9] gives support to the validity of this general picture.

The renormalization we define, following the scheme proposed in Ref. [11], is similar in spirit to the block spin transformation in statistical mechanics, in the sense that it uses a process of "elimination and rescaling". The elimination of irrelevant frequencies is done by using canonical transformations as in Kolmogorov-Arnold-Moser (KAM) theory. The frequencies we want to eliminate are the non-slow modes (non-resonant part of the perturbation), i. e. the modes which only affect the motion for a short time. The slow modes (resonant part of the perturbation), which produce the small denominators in KAM theory, are shifted towards the non-slow modes by a rescaling. They can thus be eliminated by iteration. The rescaling is the same as in Refs. [11–13]. It includes a shift of the resonances, and a rescaling of the actions and of the energy. Here and in what follows, the word "resonances" refers to the frequency vectors $\boldsymbol{\nu}_k = (p_k, q_k)$ defined by the continued fraction approximants p_k/q_k for the frequency ratio of the invariant torus. The corresponding closed orbits accumulate at the invariant torus [14], and are often used in numerical investigations.

We consider the following class of Hamiltonians with two degrees of freedom, quadratic in the action variables $\boldsymbol{A} = (A_1, A_2)$ and described by three scalar functions of the angles $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathbb{T}^2$ (the two-dimensional torus parametrized by $[0, 2\pi)^2$):

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = \frac{1}{2} m(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2 + [\boldsymbol{\omega}_0 + g(\boldsymbol{\varphi})\boldsymbol{\Omega}] \cdot \boldsymbol{A} + f(\boldsymbol{\varphi}) \ , \tag{1.1}$$

where $\boldsymbol{\omega}_0$ is the frequency vector of the considered torus, and $\boldsymbol{\Omega} = (1, \alpha)$ is some other constant vector, not parallel to $\boldsymbol{\omega}_0$.

This family of Hamiltonians has been investigated in Refs. [12,13,15]. In particular, a KAM theorem was proven for this family in Ref. [15] based on Thirring's approach [16], in which the KAM transformations are constructed such that the iteration stays within the space of Hamiltonians quadratic in the actions. In order to prove the existence of a torus with frequency vector $\boldsymbol{\omega}_0$ for Hamiltonian systems described by Eq. (1.1), it is not necessary to eliminate m, but only g and f: One has to find a canonical transformation such that the equations of motion expressed in the new coordinates show trivially the existence of this torus. For Hamiltonians (1.1), if one takes g and f equal to zero, then the resulting equations of motion are

$$\frac{d\boldsymbol{A}}{dt} = -\frac{1}{2}\frac{\partial m}{\partial \boldsymbol{\varphi}}(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2, \ \ \frac{d\boldsymbol{\varphi}}{dt} = m(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A})\boldsymbol{\Omega} + \boldsymbol{\omega}_0. \tag{1.2}$$

Then $\boldsymbol{A} = 0$ defines an invariant torus, and the motion on this torus is quasiperiodic with frequency vector $\boldsymbol{\omega}_0$

(even if the resulting Hamiltonian is not integrable).
The elimination of f and g can be achieved with canonical transformations with generating functions that are linear in the action variables, and thus map the family of Hamiltonians (1.1) into itself. This is very convenient numerically, as one only works with three scalar functions m, g and f. The only approximation involved in our numerical implementation of the transformation is a truncation of the Fourier series of these functions, e. g. we approximate

$$f(\boldsymbol{\varphi}) = \sum_{\nu \in \mathbb{Z}^2} f_\nu e^{i\nu\cdot\varphi}, \tag{1.3}$$

by

$$f^{[\leq L]}(\boldsymbol{\varphi}) = \sum_{\nu \in \mathcal{C}_L} f_\nu e^{i\nu\cdot\varphi}, \tag{1.4}$$

where $\mathcal{C}_L = \{\boldsymbol{\nu} \in \mathbb{Z}^2 \,||\nu_1| \leq L,\, |\nu_2| \leq L\}$.
In this paper, we focus on the frequency vector $\boldsymbol{\omega}_0 = (1/\gamma, -1)$, associated with the golden mean $\gamma = (1+\sqrt{5})/2$. We choose the set of frequencies I^- describing the non-slow modes as the union of the two quadrants in the plane (ν_1, ν_2) that contain the linear span of $\boldsymbol{\omega}_0$:

$$I^- = \{\boldsymbol{\nu} \in \mathbb{Z}^2 | \nu_1\nu_2 \leq 0\}. \tag{1.5}$$

The set I^- is depicted in Fig. 2.
This "frequency cutoff" restricts to Fourier modes (the non-slow modes) that can be eliminated in one renormalization step, without running into small denominator problems. As is common with cutoffs, there is not a single "natural" choice. Other possible choices of I^- will be mentioned later.

II. KAM-RENORMALIZATION-GROUP TRANSFORMATION

The renormalization scheme described in this section is for a torus of frequency vector $\boldsymbol{\omega}_0 = (1/\gamma, -1)$ where $\gamma = (1+\sqrt{5})/2$. It is straightforward to adapt it to other reduced quadratic irrationals [17].
Our transformation is composed of four steps:
(1) A canonical change of coordinates, which acts on the Fourier mode $e^{i\nu\cdot\varphi}$ for a resonance $\boldsymbol{\nu} = \boldsymbol{\nu}_k$ by a shift $\boldsymbol{\nu}_k \mapsto \boldsymbol{\nu}_{k-2}$. The resonances for $k < 2$ will be eliminated in step 4, together with all the other frequency vectors in I^-.
(2) A generalized canonical change of coordinates which corresponds to a rescaling of the action variables $\boldsymbol{A}$.
(3) A normalization corresponding to a rescaling of the energy (or time).
The composition of the three steps described so far defines a map $(m, g, f, \alpha) \mapsto (m', g', f', \alpha')$.
(4) A KAM transformation (canonical change of variables) that eliminates all non-slow modes from g' and f', i. e. the new functions f'' and g'' are in the range of $\mathbb{I}^-$, where $\mathbb{I}^-$ is a projection operator acting on a scalar function f of the angles as

$$\mathbb{I}^- f(\boldsymbol{\varphi}) = \sum_{\nu \in I^-} f_\nu e^{i\nu\cdot\varphi}. \tag{2.1}$$

Concerning step 4 we note that, for certain purposes it can be desirable to eliminate the non-slow modes from m' as well. But such a step generates terms of arbitrary order in the actions, which drastically complicates the analysis [18].
We now give a more detailed description of the four steps that define the transformation. The first step is motivated by the observation that periodic orbits for the resonant frequencies $\{\boldsymbol{\nu}_k\}$ accumulate geometrically at the (critical) $\boldsymbol{\omega}_0$–torus. This suggests that the appropriate scaling (in the angle variables) is related to a shift in the sequence of resonances. In order to implement the shift $\boldsymbol{\nu}_k \mapsto \boldsymbol{\nu}_{k-2}$ mentioned in step 1, we use the fact that the continued fraction approximants for the golden mean γ are $\gamma_k = F_{k+1}/F_k$, where F_k is the k-th Fibonacci number. These numbers can be defined recursively by the equation

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = N \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix}, \quad \text{with } N = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

starting with $F_0 = 0$ and $F_1 = 1$. In other words, if we define $\boldsymbol{\nu}_0 = (0, 1)$ then $\boldsymbol{\nu}_k = N\boldsymbol{\nu}_{k-1}$ for $k \geq 1$. Thus the desired shift $\boldsymbol{\nu}_k \mapsto \boldsymbol{\nu}_{k-2}$ is induced by the linear transformation N^{-2}, and the canonical transformation we are looking for is $(\boldsymbol{A}, \boldsymbol{\varphi}) \mapsto (N^{-2}\boldsymbol{A}, N^2\boldsymbol{\varphi})$.
As the renormalization changes the scale, some of the slow modes become non-slow modes [e. g., $(1, 1)$ is mapped into $(0, 1)$ which is an element of I^-]. We remark that there is no intrinsic sharp difference between the slow and non-slow modes. The boundary can be set at different places. We have chosen to include the coefficients $(0, 1)$ and $(1, 0)$ among the non-slow modes (I^-, to be eliminated by the KAM transformation), and $(1, 1)$ and $(2, 1)$ among the slow modes. But conceptually and practically there would be no difficulty to include e. g. $(1, 1)$ and $(2, 1)$ (or any fixed finite number of resonances) among the non-slow modes. More generally, other choices in the splitting of $\{e^{i\nu\cdot\varphi}\}$ into slow and non-slow modes should lead to the same results provided e. g. that the ratio $|\boldsymbol{\nu}|/|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}|$ is bounded on I^-, and that N^{-k} contracts vectors $\boldsymbol{\nu}$ in the complement of I^- for some fixed $k > 0$ [11].
The linear shift of the resonances multiplies $\boldsymbol{\omega}_0$ by γ^{-2} ($\boldsymbol{\omega}_0$ is an eigenvector of N); therefore we rescale the energy by a factor γ^2 in order to keep the frequency fixed at $\boldsymbol{\omega}_0$. A consequence of the shift of the resonances is that $\boldsymbol{\Omega} = (1, \alpha)$ is changed into $\boldsymbol{\Omega}' = (1, \alpha')$, where $\alpha' = (\alpha + 1)/(\alpha + 2)$.
Then we perform a rescaling of the action variables: we

change the Hamiltonian H into

$$\hat{H}(\boldsymbol{A},\boldsymbol{\varphi}) = \lambda H\left(\frac{\boldsymbol{A}}{\lambda},\boldsymbol{\varphi}\right),$$

with $\lambda = \lambda(H)$ such that the mean value of m' defined as

$$\langle m'\rangle = \int_{\mathbb{T}^2} \frac{d^2\boldsymbol{\varphi}}{(2\pi)^2} m'(\boldsymbol{\varphi}), \tag{2.3}$$

is equal to 1. Since the rescaling of energy and the shift N^2 transform the quadratic term of the Hamiltonian into $\gamma^2(2+\alpha)^2 m(\boldsymbol{\varphi})(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/2$, this condition leads to $\lambda = \gamma^2(2+\alpha)^2\langle m\rangle$.

In summary, the first three steps of our renormalization transformation rescale m, g, f and $\boldsymbol{\Omega} = (1,\alpha)$ into

$$m'(\boldsymbol{\varphi}) = \frac{m\left(N^{-2}\boldsymbol{\varphi}\right)}{\langle m\rangle}, \tag{2.4}$$

$$g'(\boldsymbol{\varphi}) = \gamma^2(2+\alpha)g\left(N^{-2}\boldsymbol{\varphi}\right), \tag{2.5}$$

$$f'(\boldsymbol{\varphi}) = \gamma^4(2+\alpha)^2\langle m\rangle f\left(N^{-2}\boldsymbol{\varphi}\right), \tag{2.6}$$

$$\alpha' = \frac{1+\alpha}{2+\alpha}. \tag{2.7}$$

We remark that the map $\boldsymbol{\Omega} \mapsto \boldsymbol{\Omega}'$ given by Eq. (2.7) has an attractive fixed point $\boldsymbol{\Omega}_* = (1,\gamma^{-1})$, which is an eigenvector of N^2, with eigenvalue $\gamma^2 > 1$.

The fourth step is carried out via an iterative procedure, similar to a Newton algorithm. We start with $H_0 = H'$. To simplify the description of the iteration step $H_n \mapsto H_{n+1}$, consider first the case where g_n and f_n depend on a (small) parameter ε_n, in such a way that $\mathbb{I}^- g_n$ and $\mathbb{I}^- f_n$ are of order $\mathcal{O}(\varepsilon_n)$. The idea is to eliminate the non-slow modes of g_n and f_n to first order in ε_n, at the expense of adding terms that are of order $\mathcal{O}(\varepsilon_n)$ in the slow modes and of order $\mathcal{O}(\varepsilon_n^2)$ in the non-slow modes.

We will work with Lie transformations $U_n : (\boldsymbol{\varphi}_{n-1},\boldsymbol{A}_{n-1}) \mapsto (\boldsymbol{\varphi}_n,\boldsymbol{A}_n)$ generated by a function S_n linear in the action variables, of the form

$$S_n(\boldsymbol{A},\boldsymbol{\varphi}) = Y_n(\boldsymbol{\varphi})\boldsymbol{\Omega}'\cdot\boldsymbol{A} + Z_n(\boldsymbol{\varphi}) + a_n\boldsymbol{\Omega}'\cdot\boldsymbol{\varphi}, \tag{2.8}$$

characterized by two scalar functions Y_n, Z_n, and a constant a_n. The expression of the Hamiltonian in the new variables is obtained by the following equation [19,20]:

$$\begin{aligned} H_{n+1} &= H_n \circ U_n = e^{+\hat{S}_n} H_n \\ &\equiv H_n + \{S_n, H_n\} + \frac{1}{2!}\{S_n,\{S_n,H_n\}\} + \cdots. \end{aligned} \tag{2.9}$$

A consequence of the linearity of S_n in $\boldsymbol{A}$ is that the Hamiltonian H_{n+1} is again quadratic in the actions, and of the form

$$\begin{aligned} H_{n+1}(\boldsymbol{A},\boldsymbol{\varphi}) &= \frac{1}{2}m_{n+1}(\boldsymbol{\varphi})(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2 \\ &+[\boldsymbol{\omega}_0 + g_{n+1}(\boldsymbol{\varphi})\boldsymbol{\Omega}']\cdot\boldsymbol{A} + f_{n+1}(\boldsymbol{\varphi}). \end{aligned} \tag{2.10}$$

We notice that the vector $\boldsymbol{\Omega}'$ remains unchanged from one step to the next. The functions Y_n, Z_n, and the constant a_n are chosen in such a way that $\mathbb{I}^- g_{n+1}$ and $\mathbb{I}^- f_{n+1}$ vanish to order $\mathcal{O}(\varepsilon_n)$; see Appendix A for details. Consequently, $\mathbb{I}^- g_{n+1}$ and $\mathbb{I}^- f_{n+1}$ are of order $\mathcal{O}(\varepsilon_0^{2^n})$, where ε_0 denotes the order of $\mathbb{I}^- g'$ and $\mathbb{I}^- f'$. If ε_0 is small, an iteration of this procedure should define a canonical transformation $U_H = U_1 \circ U_2 \circ \cdots \circ U_n \circ \cdots$ such that the Hamiltonian expressed in these new variables has only slow modes in g and f. In our numerical implementation this is indeed the case, even without small parameters.

In summary, our renormalization-group (RG) transformation acts as follows: First, some of the slow modes (resonant part of the perturbation) are turned into non-slow modes by a frequency shift and a rescaling. Then, a KAM-type iteration eliminates these non-slow (non-resonant) modes, while producing some new slow modes.

III. DETERMINATION OF THE CRITICAL COUPLING; FIXED POINT OF THE KAM-RG TRANSFORMATION

We start with the same initial Hamiltonian as in Refs. [8,12,13]

$$H(\boldsymbol{A},\boldsymbol{\varphi}) = \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A} + \varepsilon f(\boldsymbol{\varphi}), \tag{3.1}$$

where $\boldsymbol{\Omega} = (1,0)$, $\boldsymbol{\omega}_0 = (1/\gamma,-1)$, $\gamma = (1+\sqrt{5})/2$, and a perturbation

$$f(\boldsymbol{\varphi}) = \cos(\boldsymbol{\nu}_1\cdot\boldsymbol{\varphi}) + \cos(\boldsymbol{\nu}_2\cdot\boldsymbol{\varphi}), \tag{3.2}$$

where $\boldsymbol{\nu}_1 = (1,0)$ and $\boldsymbol{\nu}_2 = (1,1)$. We represent all the functions by their Fourier series truncated by retaining only the coefficients in the square $\mathcal{C}_L$ which contains $(2L+1)^2$ Fourier coefficients. For fixed L, we take successively larger couplings ε and determine whether the KAM-RG iteration converges to a Hamiltonian with $f=0$, $g=0$ (trivial fixed point), or whether it diverges ($f,g \to \infty$). By a bisection procedure, we determine the critical coupling $\varepsilon_c(L)$. We repeat the calculation with larger numbers of Fourier coefficients, to obtain a more accurate approximation. Table I lists some values of $\varepsilon_c(L)$. For instance, for $L=10$, the result is given with 7 digits; as a comparison, the method developed in Refs. [12,13] yields 4 significant digits (in order to obtain 7 significant digits with this method one needs to calculate $\varepsilon_c(L)$ up to $L=34$). Moreover, we observe the disappearance of the oscillations of Fig. 1 of Ref. [12], that is to say, the present method converges much faster than the one of Refs. [12,13].

By iterating the transformation starting from a point on the critical surface, we observe that the process converges to a nontrivial fixed point H_* (or more generally to a nontrivial fixed set related to this nontrivial fixed point by symmetries [13,21]), which we characterize by

the Fourier coefficients of the three functions m_*, g_*, f_* and $\boldsymbol{\Omega}_* = (1, \gamma^{-1})$.

Figures 3 and 4 show the weight of the Fourier coefficients of m_*. We observe that these coefficients decrease only slowly in the direction of $\boldsymbol{\omega}_0$, and in particular, along the rescaled resonances $N^{-k}\boldsymbol{\nu}_1$, $k \geq 0$, which indicates that m_* is not analytic. This is due to the fact that our transformation does not eliminate the non-slow modes of m (for the reasons mentioned earlier).

Figures 5 and 6 show the weight of the Fourier coefficients for the functions g_* and f_*, respectively. These coefficients seem to decrease exponentially, which indicates that g_* and f_* are real analytic. Notice also that, by construction, g_* and f_* have only slow modes.

At the fixed point H_*, we compute the relevant eigenvalues (critical exponents) for the linearized KAM-RG transformation. Since the breakup of invariant tori is observed in one-parameter families of Hamiltonians, indicating that the critical surface is of codimension one, one expects to find a simple eigenvalue $\delta > 1$, and no other spectrum outside the open unit disk. This is precisely what we find numerically. Table I lists the values of δ as a function of L. We obtain $\delta \in [2.650175, 2.650234]$, which is in agreement with the value obtained by MacKay for area-preserving maps [9] ($\delta = 2.650221$).

If the renormalization-group picture is correct, then all the relevant informations about critical tori with frequency vector $\boldsymbol{\omega}_0 = (1/\gamma, -1)$ is contained in the nontrivial fixed point H_*. In particular, H_* determines the observed critical scaling of phase space [5,6,8]. For the scaling factor $\lambda_* = \lambda(H_*)$, we obtain numerically $\lambda_* \in [18.827910, 18.828203]$; the value obtained for area-preserving maps is $\lambda_* = 18.828171$ (given in Refs. [5,6,9]).

IV. CONCLUSION

We have shown that in the renormalization-group approach to critical invariant tori, a partial elimination of non-resonant frequencies leads to substantial qualitative and quantitative improvements. Compared with previous schemes that have been studied numerically, our new KAM-RG transformation yields more accurate results and a better defined domain of attraction for the nontrivial fixed point. Conceptually it is close to the type of transformations used for the study of critical phenomena in statistical mechanics.

We have implemented our transformation for the case of tori with frequency vector $\boldsymbol{\omega}_0 = (1/\gamma, -1)$, where γ is the golden mean. The extension to other reduced quadratic irrationals should be straightforward and yields similar results. It also seems possible to extend our transformation to systems with three [22] and more degrees of freedom [11], where very little is known about the behavior of invariant tori under perturbations that are not necessarily small [23,24].

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APPENDIX A: ONE STEP OF THE KAM ITERATION

In this appendix, we use the notation $\boldsymbol{\partial} f = \partial f / \partial \boldsymbol{\varphi}$. We consider a Hamiltonian of the form (1.1), with $g = (1 - \mathbb{I}^-)G + \varepsilon \mathbb{I}^- G$ and $f = (1 - \mathbb{I}^-)F + \varepsilon \mathbb{I}^- F$. The parameter ε is introduced for bookkeeping purposes only; it will be set to one at the end. Our goal is to find a canonical transformation U such that the functions f' and g' for the Hamiltonian $H' = H \circ U$ have only slow modes, up to terms of order $\mathcal{O}(\varepsilon^2)$. We perform a Lie transformation defined in Sec. II. The expression of the Hamiltonian in the new variables is given by Eq. (2.9):

$$H'(\boldsymbol{A}', \boldsymbol{\varphi}') = \frac{1}{2} m'(\boldsymbol{\varphi}')(\boldsymbol{\Omega} \cdot \boldsymbol{A}')^2 + [\boldsymbol{\omega}_0 + g'(\boldsymbol{\varphi}')\boldsymbol{\Omega}] \cdot \boldsymbol{A}' + f'(\boldsymbol{\varphi}'), \tag{A1}$$

where

$$g' = g + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y + m(\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + a\Omega^2) + \mathcal{O}(\varepsilon^2), \tag{A2}$$

$$f' = f + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z + a\boldsymbol{\omega}_0 \cdot \boldsymbol{\Omega} + \mathcal{O}(\varepsilon^2), \tag{A3}$$

with Y, Z and a of order $\mathcal{O}(\varepsilon)$ to be determined. The generating function S given by Eq. (2.8) is determined by the projection of Eqs. (A2)-(A3) on the space of non-slow modes. We recall that the condition is that $\mathbb{I}^- f'$ and $\mathbb{I}^- g'$ are of order $\mathcal{O}(\varepsilon^2)$. This leads to the following equations:

$$\mathbb{I}^- f + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z = const, \tag{A4}$$

$$\mathbb{I}^- g + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y + \mathbb{I}^- (m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z) + \mathbb{I}^- m a \Omega^2 = 0. \tag{A5}$$

Equation (A5) determines a:

$$a = -\frac{\langle g \rangle + \langle m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z \rangle}{\Omega^2 \langle m \rangle}, \tag{A6}$$

Moreover the functions Y and Z have only non-slow modes, and are given by the following series:

$$Z(\boldsymbol{\varphi}) = \sum_{\nu \in I^{-*}} \frac{i}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} f_\nu e^{i\nu \cdot \varphi}, \tag{A7}$$

$$Y(\boldsymbol{\varphi}) = \sum_{\nu \in I^{-*}} \frac{i}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} \left(g_\nu + (m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z)_\nu + m_\nu a \Omega^2\right) e^{i\nu \cdot \varphi}, \tag{A8}$$

where $I^{-*} = I^- \setminus \{\mathbf{0}\}$. The transformed Hamiltonian (A1) is constructed by defining $H^{(0)} = H$ and $H^{(i)}$ for $i = 1, 2, \ldots$ by the recursive relation

$$\begin{aligned} H^{(i+1)}(\boldsymbol{A}, \boldsymbol{\varphi}) &= \{S(\boldsymbol{A}, \boldsymbol{\varphi}), H^{(i)}(\boldsymbol{A}, \boldsymbol{\varphi})\} \\ &= \frac{1}{2} m^{(i+1)}(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2 \\ &\quad + g^{(i+1)}(\boldsymbol{\varphi}) \boldsymbol{\Omega} \cdot \boldsymbol{A} + f^{(i+1)}(\boldsymbol{\varphi}), \end{aligned} \tag{A9}$$

which leads to

$$H' = \sum_{i=0}^{\infty} \frac{H^{(i)}}{i!}. \tag{A10}$$

This can be expressed in terms of the image of the three scalar functions (m, g, f) given by the following equations:

$$(m', g', f') = \left(\sum_{i=0}^{\infty} \frac{m^{(i)}}{i!}, \sum_{i=0}^{\infty} \frac{g^{(i)}}{i!}, \sum_{i=0}^{\infty} \frac{f^{(i)}}{i!} \right), \tag{A11}$$

$$(m^{(0)}, g^{(0)}, f^{(0)}) = (m, g, f), \tag{A12}$$

$$m^{(1)} = 2m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y - Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} m, \tag{A13}$$

$$\begin{aligned} g^{(1)} &= g\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y - Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} g \\ &\quad + m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + ma\Omega^2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y, \end{aligned} \tag{A14}$$

$$f^{(1)} = -Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} f + g\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + ga\Omega^2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z, \tag{A15}$$

$$m^{(i+1)} = 2m^{(i)}\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y - Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} m^{(i)}, \tag{A16}$$

$$\begin{aligned} g^{(i+1)} &= g^{(i)}\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y - Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} g^{(i)} \\ &\quad + m^{(i)}\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + m^{(i)} a\Omega^2, \end{aligned} \tag{A17}$$

$$f^{(i+1)} = -Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} f^{(i)} + g^{(i)}\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + g^{(i)} a\Omega^2, \tag{A18}$$

for $i \geq 1$.

TABLE I. Parameters associated with the breakup of noble tori.

L	Critical value $\varepsilon_c(L)$	Area multiplier	Unstable eigenvalue
	0.027590	18.828171	2.650221
3	0.027625	18.842654	2.649660
5	0.027579	18.823481	2.652722
8	0.027588	18.825034	2.649653
12	0.027590	18.829142	2.650484
13	0.027590	18.827464	2.650082
20	0.027590	18.828203	2.650234
21	0.027590	18.827772	2.650151
33	0.027590	18.828177	2.650226
34	0.027590	18.827910	2.650175

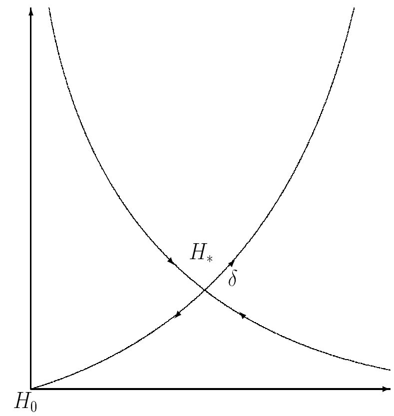

FIG. 1. Renormalization flow in the space of Hamiltonians (1.1) associated with the universal class of $\boldsymbol{\omega}_0 = (1/\gamma, -1)$.

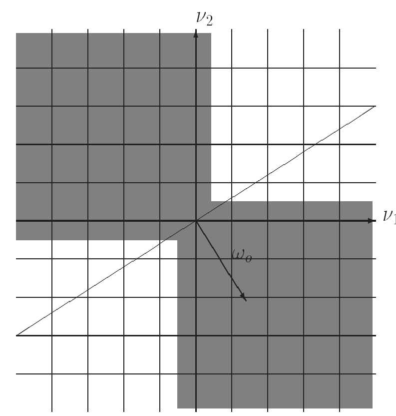

FIG. 2. Non-slow modes (in the grey part) and slow modes (in the white part) associated with $\boldsymbol{\omega}_0 = (1/\gamma, -1)$.

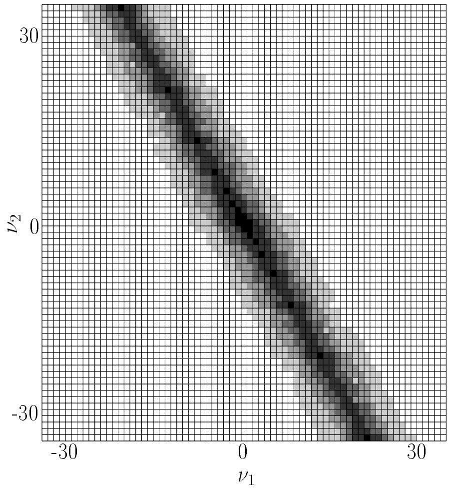

FIG. 3. Weight of the Fourier coefficients of m_*:
White: $< 10^{-10}$, grey levels: $[10^{-10}, 10^{-7}]$,
$[10^{-7}, 10^{-5}]$,$[10^{-5}, 10^{-3}]$, $[10^{-3}, 5.10^{-2}]$, black: $> 5.10^{-2}$.

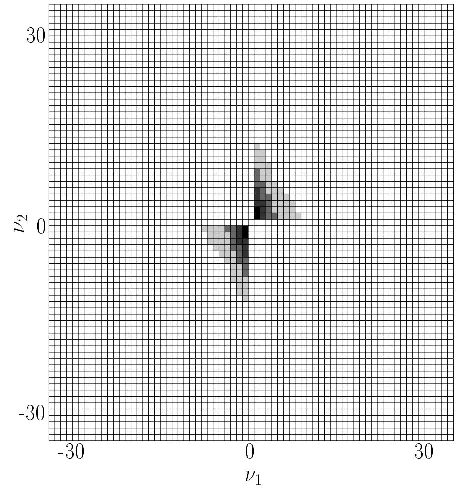

FIG. 5. Weight of the Fourier coefficients of g_*:
White: $< 10^{-10}$, grey levels: $[10^{-10}, 10^{-7}]$,
$[10^{-7}, 10^{-5}]$,$[10^{-5}, 10^{-3}]$, $[10^{-3}, 5.10^{-2}]$, black: $> 5.10^{-2}$.

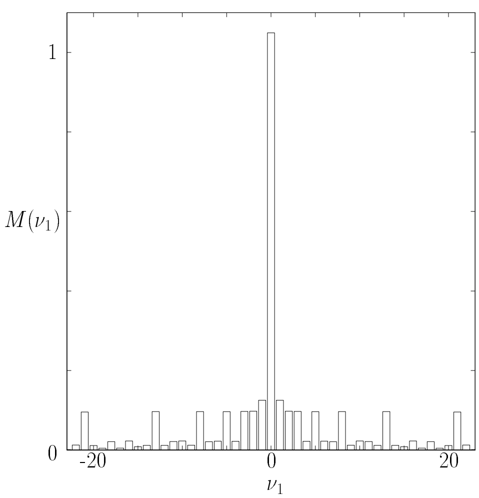

FIG. 4. Weight of the Fourier coefficients of m_*:
$M(\nu_1) = \max_{\nu_2} |m_{*\nu}|$.

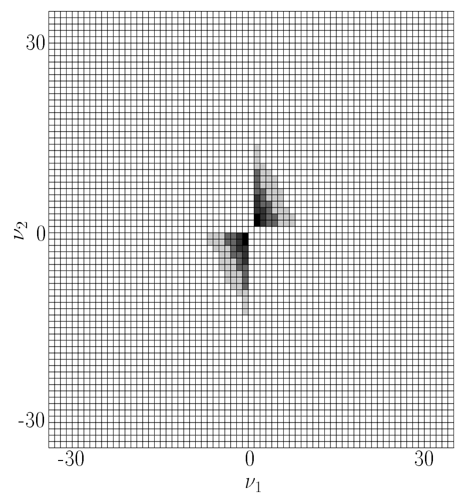

FIG. 6. Weight of the Fourier coefficients of f_*:
White: $< 10^{-10}$, grey levels: $[10^{-10}, 10^{-7}]$,
$[10^{-7}, 10^{-5}]$,$[10^{-5}, 10^{-3}]$, $[10^{-3}, 5.10^{-2}]$, black: $> 5.10^{-2}$.

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